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Some efficiencies of block designs

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Summary

Simple lower bounds for A-, D-, E- and L-efficiency of block designs are derived. The bounds are obtained on the basis of the eigenvalues of information matrix $\mathbf{C}_d$ with respect to the diagonal matrix $\mathbf{R}_d$.

Key words: A-efficiency, D-efficiency, E-efficiency, L-efficiency, block design, eigenvalues, lower bound.

1. Introduction

In some biological and industrial experiments with a small number of experimental units, the very useful balanced designs cannot be used. In practice optimal designs have to be applied, among which the A-, D-, E- and L-optimal designs deserve a particular attention. The paper deals with these experimental situations. It gives a method of assessing A-, D-, E- and L-optimal block designs. It should be noted that in the theory of experimental designs, A-, D- and E-optimality is often considered. For example, Filipiak and Szepańska (2005) and Moerbeek (2005) considered A-, D- and E-optimality for designs with quadratic and cubic growth curve models and for designs for polynomial growth models with auto-correlated errors, respectively. A-optimal chemical balance weighing designs and A-optimal designs under a quadratic growth curve model in the transformed time interval are presented respectively by Ceranka et al. (2007) and Filipiak and Szepańska (2007). The E-optimality of nested row-column designs, of designs in irregular BIB settings, of designs with three treatments and of designs under an interference model is considered by Bagchi (1996), Morgan and Reck (2007), Parvu and Morgan (2007) and Filipiak and Różański (2005), respectively.

2. Definition and notation

Any arrangement of v treatments in b blocks is called a block design d. Let $\mathbf{r}_d = (r_{d_1}, ..., r_{d_v})'$, $\mathbf{k}_d = (k_{d_1}, ..., k_{d_b})'$ and n denote a vector of treatment replications, a vector of block sizes and a number of experimental units, respectively. Let $\mathbf{R}_d$ and $\mathbf{K}_d$ be the diagonal matrices with the successive elements of $\mathbf{r}_d$ and $\mathbf{k}_d$ on their diagonals. Moreover, let $\mathbf{N}_d = (n_{d_{ij}})$ be the $v \times b$ incidence matrix, with $n_{d_{ij}}$ signifying how many times treatment i appears in block j. We assume that observations obtained in the discussed design are subject to a standard linear model

$$\mathbf{y} = \left[\mathbf{1}_n, \mathbf{D}', \boldsymbol{\Delta}'\right] \left[\mu, \boldsymbol{\beta}', \boldsymbol{\tau}'\right]' + \mathbf{e},$$

where $\mathbf{y}$ is the $n \times 1$ observation vector, $[\mathbf{1}_n, \mathbf{D}', \boldsymbol{\Delta}']$ is the $n \times (1 + b + v)$ design matrix, partitioned into $n \times 1$ vector $\mathbf{1}_n$ of unit elements, an $n \times b$ matrix $\mathbf{D}'$ whose different columns relate to different blocks, an $N \times v$ matrix $\boldsymbol{\Delta}'$ whose different columns relate to different treatments; where μ is the general parameters, $\boldsymbol{\beta}$ is the $b \times 1$ vector of block parameters, $\boldsymbol{\tau}$ is the $v \times 1$ vector of treatment parameters; and where the $n \times 1$ vector $\mathbf{e}$ of random errors has a normal distribution specified by $E(\mathbf{e}) = 0$ and $E(\mathbf{ee}') = \sigma^2 \mathbf{I}_n$. In variance analysis of experimental data and in the problem of comparison of treatments, a basic role is played by the system of normal equations for treatment parameters of the form $\mathbf{C}\boldsymbol{\tau} = \mathbf{Q}$, where $\mathbf{Q} = \mathbf{T} - \mathbf{N}\mathbf{k}^{-\delta}\mathbf{L}$ is the vector of corrected treatment sums, $\mathbf{T}$ and $\mathbf{L}$ are vectors of treatment and block sums, respectively. The information matrix for the treatment effects is known to be

$$\mathbf{C}_d = \mathbf{R}_d - \mathbf{N}_d \mathbf{K}_d^{-1} \mathbf{N}'_d. \tag{1}$$

Let μ_i $(i = 0, 1, ..., v-1)$ be eigenvalues of the matrix $\mathbf{C}_d$ and ϵ_{d_i} be eigenvalues of the matrix $\mathbf{C}_d$ with respect to the matrix $\mathbf{R}_d$, i.e.

$$\mathbf{C}_d \mathbf{p}_i = \epsilon_{d_i} \mathbf{R}_d \mathbf{p}_i \tag{2}$$

where $\mathbf{p}_i$ are eigenvectors. Assume that $0 = \epsilon_{d_0} \leq \epsilon_{d_1} \leq ... \leq \epsilon_{d_{v-1}} \leq 1$ and

$$\begin{aligned} \phi_{A/R}(d) &= \sum_{i=v-h}^{v-1} \epsilon_{d_i}^{-1}, \qquad & \phi_{D/R}(d) &= \prod_{i=v-h}^{v-1} \epsilon_{d_i}^{-1}, \\ \phi_{E/R}(d) &= \epsilon_{d_{v-h}}, \qquad & \phi_{L/R}(d) &= \sum_{i=v-h}^{v-1} \epsilon_{d_i}, \end{aligned} \tag{3}$$

where $h = r(\mathbf{C}_d)$ and $r(\mathbf{C}_d)$ denote the rank of $\mathbf{C}_d$. Note that $h \leq v-1$. If $h = v-1$ then design d is said to be connected.

The design d is A- or D-optimal if it minimizes the $\phi_{A/R}(d)$ or $\phi_{D/R}(d)$ values among all possible ones from some class of designs. A design d is E- or L-optimal if it maximizes the $\phi_{E/R}(d)$ or $\phi_{L/R}(d)$ values among all possible ones from some class of designs. The A-, D-, E- and L-efficiency of a design d is defined to be

$$\begin{aligned} e_{A/R}(d) &= \frac{\phi_{A/R}(d_A^*)}{\phi_{A/R}(d)}, & e_{D/R}(d) &= \frac{\phi_{D/R}(d_D^*)}{\phi_{D/R}(d)}, \\ e_{E/R}(d) &= \frac{\phi_{E/R}(d)}{\phi_{E/R}(d_E^*)}, & e_{L/R}(d) &= \frac{\phi_{L/R}(d)}{\phi_{L/R}(d_L^*)}, \end{aligned} \tag{4}$$

where d_A^*, d_D^*, d_E^* and d_L^* are A-, D-, E- and L-optimal designs, respectively.

It is worth mentioning that $0 \leq e_{A/R}(d) \leq 1$, $0 \leq e_{D/R}(d) \leq 1$, $0 \leq e_{E/R}(d) \leq 1$ and $0 \leq e_{L/R}(d) \leq 1$ therefore if $e_{A/R}(d) = 1$ or $e_{D/R}(d) = 1$ or $e_{E/R}(d) = 1$ or $e_{L/R}(d) = 1$ then design d is A-, D-, E- and L-optimal, respectively.

One problem with the above definitions is that optimal designs are known only for some special cases. Therefore, in the next section lower bounds of $e_{A/R}$, $e_{D/R}$, $e_{E/R}$ and $e_{L/R}$ will be given for appraising the efficiencies of design d. It should be noted that functions (3) and efficiencies (4), where $\mathbf{C}_d\mathbf{p}_i = \epsilon_{d_i}\mathbf{p}_i$, are given by Brzeskwiniewicz (1996).

3. Results

3.1. Lower bounds of $e_{A/R}$ and $e_{D/R}$

Let $\Omega_{n,v,b,k_{max},h}$ denote the set of all block designs with the some parameters $(n, v, b, k_{\max}, h)$, where $k_{\max} = \max\{k_{d_j} : \ j = 1, ..., b\}$. From (2) we have $\left(\mathbf{R}_d{}^{-1}\mathbf{C}_d\right)\mathbf{p}_i = \epsilon_{d_i}\mathbf{p}_i$ which implies that ϵ_{d_i} are eigenvalues of matrix $\mathbf{A}_d = \mathbf{R}_d{}^{-1}\mathbf{C}_d$. Note that

$$\begin{aligned} tr(\mathbf{A}_d) &= \sum_{i=1}^{v}\left(1 - \sum_{j=1}^{b}\frac{n_{d_{ij}}^2}{r_{d_i}k_{d_j}}\right) \leq v - \sum_{i=1}^{v}\frac{1}{r_{d_i}k_{\max}}\sum_{j=1}^{b}n_{d_{ij}} = \\ &= v - \sum_{i=1}^{v}\frac{1}{k_{\max}} = \frac{v(k_{\max}-1)}{k_{\max}} \end{aligned} \tag{5}$$

and

$$\bar{\epsilon}_{d_i} = \frac{\sum_{i=v-h}^{v-1} \epsilon_{d_i}}{h} \leq \frac{v\,(k_{\max}-1)}{hk_{\max}}. \tag{6}$$

Observe that

$$\sum_{i=v-h}^{v-1} \epsilon_{d_i}^{-1} \geq \frac{h}{\bar{\epsilon}_{d_i}} \qquad \text{and} \qquad \prod_{i=v-h}^{v-1} \epsilon_{d_i}^{-1} \geq \left(\frac{1}{\bar{\epsilon}_{d_i}}\right)^h. \tag{7}$$

From (3), (5) and (6) we have $\phi_{A/R}\,(d_A^*) \geq \frac{h^2 k_{max}}{v(k_{max}-1)}$ and $\phi_{D/R}\,(d_D^*) \geq \left(\frac{hk_{max}}{v(k_{max}-1)}\right)^h$. From these inequalities and from (4) it follows that $e_{A/R}\,(d) \geq \frac{h^2 k_{max}}{v(k_{max}-1)\phi_{A/R}(d)}$ and $e_{D/R}\,(d) \geq \left(\frac{hk_{max}}{v(k_{max}-1)}\right)^h \cdot \frac{1}{\phi_{D/R}(d)}$, and consequently two efficiency lower bounds of $e_{A/R}$ and $e_{D/R}$ are defined as

$$\begin{aligned} e'_{A/R}\,(d) &= \tfrac{h^2 k_{max}}{v(k_{max}-1)\phi_{A/R}(d)}, \\ e'_{D/R}\,(d) &= \left(\tfrac{hk_{max}}{v(k_{max}-1)}\right)^h \tfrac{1}{\phi_{D/R}(d)}. \end{aligned} \tag{8}$$

3.2. Lower bounds of $e_{E/R}$

From Brzeskwiniewicz (1989) it follows that if design d contains a block which consists of m treatments and $2 \leq m \leq v-1$, then $\mu_{d_1} \leq P_{C_d}\,(m)$ and $\mu_{d_1} \leq T_{C_d}$, where μ_{d_i} denotes the eigenvalues of C_d with $0 = \mu_{d_0} < \mu_{d_1} \leq ... \leq \mu_{d_{v-1}}$

$$\begin{aligned} P_{C_d}\,(m) &= \frac{v}{m(v-m)} \cdot \frac{\sum_{i=1}^{m} r_{d_i}\,(k_{\max}-1) - k_{\max}\,(k_{d_1}-1)}{k_{\max}}, \\ T_{C_d} &= \frac{v}{v-1} \cdot \frac{r_{\min}\,(k_{\max}-1)}{k_{\max}}, \end{aligned} \tag{9}$$

where $r_{\max} = \max\{r_{d_1}, ..., r_{d_m}\}$ and $r_{\min} = \min\{r_{d_1}, ..., r_{d_m}\}$.

In the above it is assumed, possibly by relabelling the treatments and reshuffling the blocks, that the first block in design d consists of m treatments with numbers $1, ..., m$. Note that a proof of (9) if $r_{d_1} = r_{d_2} = ... =$

$r_{d_v} = r_d$ and $k_{d_1} = k_{d_2} = ... = k_{d_b} = k_d$, is also given by Constantine (1982). From (9) we have

$$\epsilon_{d_1} \leq \frac{P_{C_d}(m)}{r_{\min}} \leq P_d(m) \quad \text{and} \quad \epsilon_{d_1} \leq \frac{T_{C_d}(m)}{r_{\min}} = T_d, \tag{10}$$

where $P_d(m) = \frac{v}{m(v-m)} \cdot \frac{mr_{\max}(k_{\max}-1)-k_{\max}\left(k_{d_1}-1\right)}{k_{\max}r_{\min}}$ and $T_d = \frac{v}{(v-1)} \cdot \frac{k\max-1}{k_{\max}}$. Observe that from (10) we have

$$\epsilon_{d_1} \leq \min\{P_d(m), T_d\}. \tag{11}$$

From (3) and (11) it follows that

$$\phi_{E/R}\,(d_E^*) \leq \min\{P_d(m), T_d\} \tag{12}$$

which with (4) leads to

$$e_{E/R}\,(d) \geq \frac{\phi_{E/R}\,(d)}{\min\{P_d(m), T_d\}}. \tag{13}$$

From the above formula the lower bound of $e_{E/R}$ is defined by

$$e'_{E/R}\,(d) = \frac{\phi_{E/R}\,(d)}{\min\{P_d(m), T_d\}}. \tag{14}$$

Note that examples of block designs with very high E-efficiency are given by Brzeskwiniewicz and Krzyszkowska (2007).

3.3. Lower bounds of $e_{L/R}$

From (3) and (5) we have

$$\phi_{L/R}\,(d_L^*) \leq \frac{v\,(k_{\max}-1)}{k_{\max}}. \tag{15}$$

The above formula and (4) imply that

$$e_{L/R}\,(d) \geq \frac{k_{\max}\phi_{L/R}(d)}{v\,(k_{\max}-1)} \tag{16}$$

and consequently the lower bound of $e_{L/R}$ is defined by

$$e'_{L/R}\,(d) = \frac{k_{\max}\phi_{L/R}(d)}{v\,(k_{\max}-1)}. \tag{17}$$

4. Conclusion

Note that $0 \leq e'_{A/R}(d) \leq 1$, $0 \leq e'_{D/R}(d) \leq 1$, $0 \leq e'_{E/R}(d) \leq 1$, $0 \leq e'_{L/R}(d) \leq 1$. If $e'_{A/R}(d) = 1$ or $e'_{D/R}(d) = 1$ or $e'_{E/R}(d) = 1$ or $e'_{L/R}(d) = 1$ then design d is A-, D-, E- or L-optimal, respectively which follows from (4). When the design d gives $e'_{A/R}(d)$ (or $e'_{D/R}(d)$ or $e'_{E/R}(d)$ or $e'_{L/R}(d)$) close to one, then one can say that its A (or D or E or L) optimality is high. And on the other hand, a value for $e'_{A/R}(d)$ (or $e'_{D/R}(d)$or $e'_{E/R}(d)$ or $e'_{L/R}(d)$) close to zero indicates that the discussed design is far from being an A-optimal (or D-optimal or E-optimal or L-optimal) one.

5. Examples

We consider A-, D-, E-, L-efficiency of the following block designs:

(i)

$i \backslash j$	1	2	3	4	5	6	r_{d_i}
1	4	4	3	3	3	3	20
2	1	0	0	1	1	1	4
3	1	0	1	0	1	1	4
4	0	1	1	1	0	1	4
5	0	1	1	1	1	0	4
k_{d_j}	6	6	6	6	6	6	36

(ii)

$i \backslash j$	1	2	3	4	5	6	r_{d_i}
1	1	1	1	1	1	1	6
2	1	1	1	0	0	0	3
3	1	1	0	0	0	0	2
4	1	0	1	0	0	0	2
k_{d_j}	4	3	3	1	1	1	13

In case (i) $d \in \Omega_{36,5,6,6,4}$ and $\epsilon_{d_0} = 0$, $\epsilon_{d_1} = \frac{105}{120}$, $\epsilon_{d_2} = \epsilon_{d_3} = \frac{115}{120}$, $\epsilon_{d_4} = \frac{117}{120}$. Firstly, we calculate $\phi_{\cdot/R}(d)$, T_d and $P_d(3)$ occurring respectively in (3), (10) and (9) as: $\phi_{A/R}(d) = 120\left(\frac{1}{105} + \frac{2}{115} + \frac{1}{117}\right) = 4.26$, $\phi_{D/R}(d) = \frac{120}{105} \cdot \left(\frac{120}{115}\right)^2 \cdot \frac{120}{117} = 1.28$, $\phi_{E/R}(d) = \frac{105}{120} = 0.875$, $\phi_{L/R}(d) = \frac{1}{120}(105 + 2 \cdot 115 + 117) = \frac{452}{120} = 3.77$, $T_d = \frac{5}{4} \cdot \frac{5}{6} = 1.04$, $P_d(3) = \frac{5}{3 \cdot 2} \cdot \frac{3 \cdot 20 \cdot 5 - 6 \cdot 5}{6 \cdot 4} = 9.375$. Hence according to formulae (8), (14) and (17) we obtain: $e'_{A/R}(d) = \frac{4^2 \cdot 6}{5 \cdot 5 \cdot 4.26} = 0.90$, $e'_{D/R}(d) = \left(\frac{4 \cdot 6}{5 \cdot 5}\right)^4 \cdot \frac{1}{1.28} = 0.66$, $e'_{E/R}(d) = \frac{0.875}{\min\{1.04, 9.375\}} = 0.84$, $e'_{L/R}(d) = \frac{6 \cdot 3.77}{5 \cdot 5} = 0.90$. We can say that the A-, D-, E- and L-efficiency of the design (i) are high.

In case (ii) $d \in \Omega_{13,4,6,4,3}$ and $\epsilon_{d_0} = 0$, $\epsilon_{d_1} = 0.83$, $\epsilon_{d_2} = 0.98$, $\epsilon_{d_3} = 0.64$. According to formulae as in case (i) we obtain: $\phi_{A/R}(d) = 3.77$, $\phi_{D/R}(d) = 1.90$, $\phi_{E/R}(d) = 0.83$, $\phi_{L/R}(d) = 2.46$, $T_d = 1$, $P_d(3) = 7$, $e'_{A/R}(d) = 0.80$, $e'_{D/R}(d) = 0.53$, $e'_{E/R}(d) = 0.83$, $e'_{L/R}(d) = 0.82$. We can say that the A-, D-, E- and L-efficiency of the design (ii) are high.

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